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Deformations of Cones

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1. Deformations of isolated singularities.

Let X be an affine scheme of finite type over a field k, A a ring with an augumentation $A \to k$. By a <u>deformation</u> X_A of X over A, we mean a product diagram

$$\begin{array}{ccc} X_A & \leftarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}\ A & \leftarrow & \mathrm{Spec}\ k \end{array} \qquad X \simeq X_A \otimes_A k,$$

where X_A is <u>flat</u> over $\mathrm{Spec}\ A$. If X'_A is another deformation of X over A, then X_A and X'_A are <u>isomorphic</u> if there exists an isomorphism $f: X_A \to X'_A$ over A which induces the identity over $\mathrm{Spec}\ k$.

Throughout this lecture, we assume the following:

i) X has only isolated singularities.

ii) A is an artinian local k-algebra with residue field k (e.g., $A = k[t]/t^n$ etc.).

We write $X = \mathrm{Spec}\ B$ with $B = k[x]/(f_1, \ldots, f_q)$ where $x = (x_1, \ldots, x_N)$ is a set of variables and $f_i \in k[x]$. For simplicity of notation, we set $P = k[x]$ and

$F = (f_1, \ldots, f_q) P$. For some integer p, we have a resolution

$$(1) \qquad P^p \xrightarrow{R} P^q \xrightarrow{f} P \longrightarrow B \longrightarrow 0$$

of B.

We note that any deformation X_A of X over A is an affine scheme ([EGA I.5.1.9]). So we can write $X_A = \operatorname{Spec} B_A$ with an A-algebra B_A such that $B_A \otimes_A k = B$. Moreover, X_A can be embedded in $\mathbb{A}_A^N = \operatorname{Spec} P_A$ where $P_A = A[x]$. By an <u>embedded deformation</u> of X (with respect to $\mathbb{A}^N = \operatorname{Spec} P$) over A, we mean a closed subscheme X_A of $\mathbb{A}_A^N$ flat over $\operatorname{Spec} A$ which induces the subscheme X of $\mathbb{A}^N$.

Proposition 1. <u>The resolution</u> (1) <u>lifts to a resolution</u>

$$(2) \qquad P_A^p \xrightarrow{R_A} P_A^q \xrightarrow{f_A} P_A \longrightarrow B_A \longrightarrow 0.$$

In fact, the assertion is equivalent to the flatness of X_A over $\operatorname{Spec} A$

We consider the following three deformation functors:

Defs(X) : $A \longrightarrow$ the set of isomorphic classes of deformations of X over A.

Emb.Defs(X) : $A \longrightarrow$ the set of embedded deformations of X over A.

Defs of Res : $A \longrightarrow$ the set of isomorphic classes of liftings (2) of (1).

Then we have three natural morphisms of functors:

$$
\begin{tikzcd}
 & \mathrm{Emb.Defs}(X) \arrow[dr] & \\
\text{Defs of Res} \arrow[ur] \arrow[rr] & & \mathrm{Defs}(X) .
\end{tikzcd}
$$

Corollary. <u>These morphisms are smooth</u> (see [F], Definition 2.2).

2. The tangent space of Defs(X).

Let $A = k[t]/t^2$ where t is a variable. The set of isomorphic classes of deformations of X over A is called the <u>tangent</u> <u>space</u> of the functor Defs(X). A deformation X_A of X over A is defined by $f_{A,i} \varepsilon P_A$ $(1 \leqq i \leqq q)$. We write

$$f_{A,i} = f_i + g_i t \qquad \text{with } g_i \varepsilon P.$$

The flatness of X_A over Spec A is equivalent to the existence of an A-valued $p \times q$-matrix R_A such that $R_A \equiv R \mod (t)$ and $f_A R_A = 0$, where f_A denote the vector $(f_{A,1}, \ldots, f_{A,q})$. If we write $R_A = R + St$ with a k-valued $p \times q$-matrix S, then the above condition is equivalent to $gR + fS = 0$. Hence, we have

$f_A = f + gt$ defines a deformation of X.

$\Longleftrightarrow$ $gR \equiv 0 \mod F$.

$\Longleftrightarrow$ g defines a P-homomorphism $P^q/RP^p = F \longrightarrow B$.

$\Longleftrightarrow$ g defines a B-homomorphism $F/F^2 \longrightarrow B$.

We define the normal sheaf by

$$N_B = N_X = \mathrm{Hom}_B(F/F^2, B).$$

Then the tangent space of Emb. Defs(X) is isomorphic to N_B.

Next we shall kill the effect of automorphisms of $\mathbb{A}^N_A$. Let h be an automorphism of $\mathbb{A}^N_A$ over A which induces the identity on $\mathbb{A}^N_k$. Then h corresponds to $\tilde{h}: P_A \to P_A$ given by $\tilde{h}(x_i) = x_i + y_i t$ with $y_i \in P$. Hence $h^{-1}(X_A)$ is defined by $f + g't = f_A(x+yt)$. If we let J denote the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$, then we have $g' = g + y^t J$.

Letting Θ_X and $\Theta_{\mathbb{A}}$ denote the sheaf of k-derivations of X and $\mathbb{A}^N$, respectively, we have the exact sequence

$$0 \longrightarrow \Theta_X \xrightarrow{J} \Theta_{\mathbb{A}|X} \longrightarrow N_X.$$

We set $T^1_X = \mathrm{Coker}\,(\Theta_{\mathbb{A}|X} \longrightarrow N_X)$. Then the tangent space of Defs(X) is isomorphic to T^1_X. We note that the support of T^1_X is concentrated at the singular locus of X.

For example, assume that X is normal and that $\dim X \geq 2$. We set $U = X -$ (singular locus). Since $T^1_X = 0$ on U, we have an exact sequence

$$(*) \qquad H^0(U, \Theta_X) \longrightarrow H^0(U, \Theta_{\mathbb{A}|U}) \longrightarrow H^0(U, N_X) \longrightarrow H^1(U, \Theta_X).$$

Since depth $\mathcal{O}_{X,x} \geq 2$ for any $x \in X$, the restriction maps $H^0(X, \Theta_X) \longrightarrow H^0(U, \Theta_U)$ and $H^0(X, \Theta_{\mathbb{A}}|_X) \longrightarrow H^0(U, \Theta_{\mathbb{A}}|_U)$ are bijective. Hence we can identify T^1_X with a subspace of Coker $(H^0(U, \Theta_{\mathbb{A}}|_U) \longrightarrow H^0(U, N_X))$.

Theorem 1. (Schlessinger) _If_ X _is an affine scheme over a field_ k _with only isolated singularities, then there exists a formal versal deformation of_ X _parametrized by a complete local ring_ $\hat{R}$.

For the proof, see [F], Proposition 3.10. A formal versal deformation of X means a _hull_ of the functor Defs(X). By the definition of a hull, if we let m denote the maximal ideal of $\hat{R}$, we have

$$m/m^2 \simeq \text{tangent space of Defs}(X) \simeq T^1_X.$$

Problem. _Compute_ $\hat{R}$.

We say that X is _unobstructed_, if the functor Defs(X) is smooth, or equivalently, if $\hat{R}$ is a formal power series ring.

In the following cases, X is unobstructed.

(i) X is a complete intersection.

(ii) (Schaps) X is a Cohen-Macaulay subscheme of codimension 2 in an affine space.

3. Deformations of cones.

Let Y be a smooth subscheme in $\mathbb{P}^m$, C the cone over Y in $\mathbb{A}^{m+1}$, and $\overline{C} = C \cup Y_\infty$ the cone over Y in $\mathbb{P}^{m+1}$. We ask to relate deformations of C, Y, and $\overline{C}$. We assume that Y is arithmetically normal (i.e., the systems of hypersurfaces of any degree are complete), or equivalently, C is normal. Let v be the vertex of C and $\overline{C}$, $U = C - v$, and $\overline{U} = \overline{C} - v$. We set

$$L = \mathbb{P}^{m+1} - v = \text{line bundle } O_{\mathbb{P}^{m+1}}(1),$$

$$V = L - (0\text{-section}).$$

Then $L = \underline{\mathrm{Spec}}\ S(O(-1)) = \underline{\mathrm{Spec}} \bigoplus_{n=-\infty}^{0} O(n)$ and $V = \underline{\mathrm{Spec}} \bigoplus_{n=-\infty}^{\infty} O(n)$ is the $\mathbb{G}_m$-bundle associated to L.

For any sheaf M on $\mathbb{P}^m$, we have

$$H^q(L, \pi^*M) = \bigoplus_{n=-\infty}^{0} H^q(\mathbb{P}^m, M(n)),$$

$$H^q(V, \pi^*M) = \bigoplus_{n=-\infty}^{\infty} H^q(\mathbb{P}^m, M(n)),$$

where π denote the natural projections onto $\mathbb{P}^m$.

Letting $\Theta_{X/Y}$ denote the sheaf of derivations of X over Y, we have the standard exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \Theta_{V/\mathbb{P}^m} \arrow[r] \arrow[d, "\wr"] & \Theta_V \arrow[r] \arrow[d, "\wr"] & \pi^*\Theta_{\mathbb{P}^m} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \pi^*O_{\mathbb{P}^m} \arrow[r] & \pi^*O(1)^{m+1} \arrow[r] & \pi^*\Theta_{\mathbb{P}^m} \arrow[r] & 0
\end{tikzcd}$$

Moreover, letting N_U and N_Y denote the normal sheaves of U and Y in V and $\mathbb{P}^m$, respectively, we have a commutative diagram

$$
(**)\qquad
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^*O_Y & \longrightarrow & \Theta_U & \longrightarrow & \pi^*\Theta_Y & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^*O_Y & \longrightarrow & \Theta_{V|U} & \longrightarrow & \pi^*\Theta_{\mathbb{P}^m|Y} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N_U & \xrightarrow{\sim} & \pi^*N_Y & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where all vertical and horizontal lines are exact. We note that the sequence (*) is derived from the second vertical line in (**). Thus we get a commutative diagram

$$
\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
H^0(C,\ \Theta_{V|C}) & \longrightarrow & H^0(C,\ N_C) & \longrightarrow & H^0(C,\ T^1_C) & \longrightarrow & 0 \\
\downarrow\wr & & \downarrow\wr & & \downarrow & & \\
H^0(U,\ \Theta_{V|U}) & \longrightarrow & H^0(U,\ N_U) & \longrightarrow & H^1(U,\ \Theta_U) & & \\
\downarrow\wr & & \downarrow\wr & & & & \\
\bigoplus_{n=-\infty}^{\infty} H^0(Y, O(n+1))^{m+1} & \longrightarrow & \bigoplus_{n=-\infty}^{\infty} H^0(Y,\ N_Y(n)) & & & &
\end{array}
$$

This shows that T^1_C is graded as $\bigoplus_{n=-\infty}^{\infty} T^1_C(n)$.

Let $\mathrm{Hilb}(\overline{C})$ and $\mathrm{Hilb}(Y)$ denote the Hilbert functors of $\overline{C}$ and Y in $\mathbb{P}^{m+1}$ and $\mathbb{P}^m$, respectively.

Theorem 2. (1) (Pinkham) <u>If</u> $T^1_C(n) = 0$ <u>for</u> $n > 0$, <u>then the natural morphism</u>

$$\mathrm{Hilb}(\overline{C}) \longrightarrow \mathrm{Defs}(C)$$

is smooth.

(2) (Schlessinger) <u>If</u> $T^1_C(n) = 0$ <u>for</u> $n \neq 0$, <u>then every deformation of</u> C <u>is a cone</u>, <u>namely the natural morphism</u>

$$\mathrm{Hilb}(Y) \longrightarrow \mathrm{Defs}(C)$$

<u>is smooth</u>.

Proof. (1) It suffices to prove the following:

(i) The tangent map is surjective.

(ii) Let $A' \longrightarrow A$ be a small extension of artinian local k-algebras with residue field k, $C_A \varepsilon \mathrm{Defs}(C)_{/A}$ (= A-valued point of Defs(C)), $C_{A'} \varepsilon \mathrm{Defs}(C)_{/A'}$, and $\overline{C}_A \varepsilon \mathrm{Hilb}(\overline{C})_{/A}$ such that $C_{A'}$ induces C_A over Spec A and $\overline{C}_A$ induces C_A on C. Then we can find $\overline{C}_{A'} \varepsilon \mathrm{Hilb}(\overline{C})_{/A'}$ which induces $\overline{C}_A$ over Spec A.(see [F], Definition 2.2).

In order to prove (i), we note that

$$H^0(\overline{C}, N_{\overline{C}}) \simeq H^0(\overline{U}, N_{\overline{U}}) = \bigoplus_{n=-\infty}^{0} H^0(Y, N_Y(n)) ,$$

and that

$$\bigoplus_{n=-\infty}^{\infty} H^0(Y, N_Y(n)) = H^0(U, N_U) \longrightarrow T^1_C$$

is surjective. Hence the hypothesis implies that the tangent map $H^0(\overline{C}, N_{\overline{C}}) \longrightarrow T^1_C$ is surjective.

(ii) In general, the obstruction ξ for extending $\overline{C}_A$ to $\overline{C}_{A'}$ lies in $H^1(\overline{C}, N_{\overline{C}})$. Since we have depth $O_{C,z} \geqq 2$ for any point $z \in C$, we have an injection

$$\begin{array}{ccccc} H^1(\overline{C}, N_{\overline{C}}) & \hookrightarrow & H^1(\overline{U}, N_{\overline{U}}) & = & \bigoplus_{n=-\infty}^{0} H^1(Y, N_Y(n)) \\ & & & & \downarrow \\ & & H^1(U, N_U) & = & \bigoplus_{n=-\infty}^{\infty} H^1(Y, N_Y(n)) . \end{array}$$

The existence of $C_{A'}$ implies that the image of ξ in $H^1(U, N_U)$ is zero. This proves the existence of $\overline{C}_{A'}$.

The proof of (2) is similar, so we omit it. q.e.d.

Corollary. _In the above situation, assume that_ $\dim Y \geqq 2$ _and that_ $O_Y(1)$ _is sufficiently ample in the sense that_

$$H^1(Y, O_Y(n)) = 0 \quad \text{for} \quad n \neq 0,$$

$$H^1(Y, \Theta_Y(n)) = 0 \quad \text{for} \quad n \neq 0.$$

Then every deformation of C _is a cone_.

Finally we study the special case: Y is a rational curve in $\mathbb{P}^m$ of degree m with a generic point $(1, t, t^2, \ldots, t^m)$.

Y is defined by

$$\operatorname{rank} \begin{pmatrix} x_0 & x_1 & \cdots\cdots & x_{m-1} \\ x_1 & x_2 & \cdots\cdots & x_m \end{pmatrix} < 2.$$

Theorem 3. _If_ Y _is a rational curve in_ $\mathbb{P}^m$ _of degree_ m, _then the natural morphism_

$$\mathrm{Hilb}(\bar{C}) \longrightarrow \mathrm{Defs}(C)$$

is smooth.

This follows from $H^1(Y, \mathcal{O}_Y(n)) = H^1(Y, \Theta_Y(n)) = 0$ for $n > 0$.

We refer to the result of Nagata [N]: Let X be a surface in $\mathbb{P}^{m+1}$ of degree m.

(1) If X is singular, X is a cone over the singular point.

(2) If X is non-singular, X is a scroll, i.e., a rational ruled surface embedded linearly on fibres, unless $m = 4$ and X is the Veronese embedding of $\mathbb{P}^2$.

Using this fact, Pinkham proved

Corollary. Let $M = \mathrm{Spf}(\hat{R})$ be the parameter space of a versal deformation of C.

i) For $m = 2$ or 3, M is smooth.

ii) For $m = 4$, M has two components of dimension 3 and 1, which correspond to scrolls and Veronese embeddings of $\mathbb{P}^2$, respectively.

iii) For $m > 4$, M_{red} is smooth of dimension $m - 1$, but $\hat{R}$ has non-zero nilpotent elements.

(Notes by E. Horikawa)

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